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Quasi-Stationary Regime of a Branching Random Walk in Presence of an Absorbing Wall

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Abstract A branching random walk in presence of an absorbing wall moving at a constant velocity v undergoes a phase transition as the velocity v of the wall varies. Below the critical velocity v_c, the population has a non-zero survival probability and when the population survives its size grows exponentially. We investigate the histories of the population conditioned on having a single survivor at some final time T. We study the quasi-stationary regime for $v < v_c$ when T is large. To do so, one can construct a modified stochastic process which is equivalent to the original process conditioned on having a single survivor at final time T. We then use this construction to show that the properties of the quasi-stationary regime are universal when $v \to v_c$. We also solve exactly a simple version of the problem, the exponential model, for which the study of the quasi-stationary regime can be reduced to the analysis of a single one-dimensional map.

Keywords Branching random walks · Quasi-stationary regime · Traveling waves · Birth-death processes · Transition to an absorbing state

1 Introduction

Branching random walks are often used as simple models of evolving populations, with or without selection. They can describe how a population invades an empty domain, how a favorable mutation spreads [19, 20] or how the fitness of a population evolves under selection [16, 29]. Recently very simple models [7, 9, 29, 41] of neo-Darwinian evolution [39] have been studied, motivated in particular by in vitro experiments [16]. In these models each individual of a population of fixed size is characterized by a single number, a trait, which we

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can call its fitness or its adaptability. From one generation to the next, this trait is transmitted to the offspring up to small variations which represent the effect of mutations. Then at each generation, only the N fittest individuals survive. It was shown in [9] that these models can be mathematically reduced to the study of traveling waves in presence of noise, a problem [14, 17, 32–34, 37, 38] which appears in many other contexts such as reaction-diffusion problems, disordered systems [6] or QCD [25, 30, 35].

Another version of the same model of evolution was also considered in [12] where the size of the population may vary with time and the selection takes place by eliminating at each generation all the individuals which are below some threshold along the fitness axis. When this threshold increases linearly with time, the model reduces to the problem of a branching random walk in presence of an absorbing wall moving at a constant velocity v. Depending on how fast the wall moves, the population can either survive or get extinct. The survival probability, in the long time limit, decreases as the velocity v of the wall increases and, above some critical velocity v_c, the population gets extinct with probability 1.

A model fully equivalent to the branching random walk in presence of an absorbing moving wall, that we study here, is the case of a branching random walk with a drift and a fixed absorbing threshold, which has been introduced in [1] in order to describe the aging process in a cell proliferation model. In this case too, there is a phase transition to an absorbing state as the drift increases.

Similar phase transitions to an absorbing state (the empty population) have been extensively studied in the theory of non-equilibrium systems [24, 36], in problems like directed percolation [15] or reaction-diffusion processes [26]. Beyond the characterization of the transition to this absorbing state, there has been an increasing interest to study the process conditioned on the survival at a very late time [11, 13, 22].

In many cases, evolutions conditioned to avoid the absorbing empty state lead to a quasi-stationary regime: for example, in a birth-death process (see [2, 10, 23, 27, 28] and Sect. 2 below), if one conditions on a given non-zero size at a final time T and looks at the population at a time t such that both t and $T-t$ are much larger than the characteristic relaxation time of the system, the statistical properties depend neither on the initial condition nor on t and T and one observes a quasi-stationary state.[1] The probability of the events which contribute to these quasi-stationary regimes decreases with T, and therefore they are difficult to observe in simulations. There are however a number of cases for which one can modify the process (i.e. determine the right bias) [21, 22] to generate the quasi-stationary regime directly.

In the present work, we consider the problem of a branching random walk in presence of an absorbing boundary which moves at a constant velocity v (see Fig. 1 and [12]). On the infinite line, i.e. without any boundary, the population spreads linearly in time [5, 31] at a velocity v_c and the number of walkers grows exponentially with time. The presence of a moving absorbing wall introduces a competition between the growth and the extinction due to the wall. When $v > v_c$, the population gets extinct with probability 1, whereas when $v < v_c$, this extinction probability is strictly less than 1 and there is a non-zero probability that the population grows exponentially with time. Here, we mostly study the quasi-stationary regime, conditioned on the survival of a few individuals at a very late time T,

[1] In the present paper, we use the term "quasi-stationary" to describe a situation at time t conditioned on the fact that there is a fixed non-zero number of individuals at a much larger time T. Note that "quasi-stationary" is often used to describe a different situation where the two times t and T coincide [13, 18, 42, 45]. What we call here a quasi-stationary process is sometimes called a "Q-process" [10].

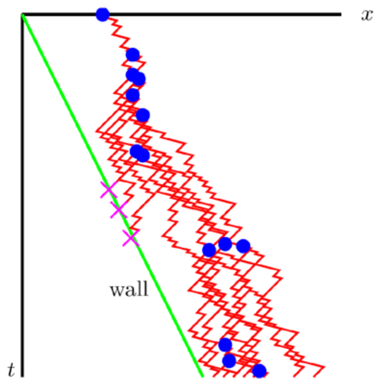


Fig. 1 Branching random walk in presence of an absorbing wall. The *circles* represent branching events and the *crosses* absorption by the wall

for $v < v_c$. This quasi-stationary regime has already been studied in the mathematical literature, where a modified process was invented [21, 22] which allows one to generate typical evolutions in this quasi-stationary regime. Our main goal in the present paper is to show how the properties (average size of the population, density profiles) can be obtained in the quasi-stationary regime and how universal expressions for these properties emerge when $v \to v_c$.

The paper is organized as follows: in Sect. 2, we recall how the quasi-stationary regime can be calculated in the case of a simple birth-death process, and how the distribution of the population size becomes universal near the transition. In Sect. 3, we extend the approach of Sect. 2 to the branching random walk process in presence of an absorbing wall. In Sect. 4, we show how the quasi-stationary regime can be generated by a modified process, as already described in [21, 22] and we obtain explicit expressions of the density in this regime. In Sect. 5, we show that the average size of the population and the average density profile become universal as $v \to v_c$. In Sect. 6, we analyze a simpler model, the exponential model, for which the whole distribution of the population size can be calculated.

2 An Example of Birth-Death Processes: The Galton-Watson Process

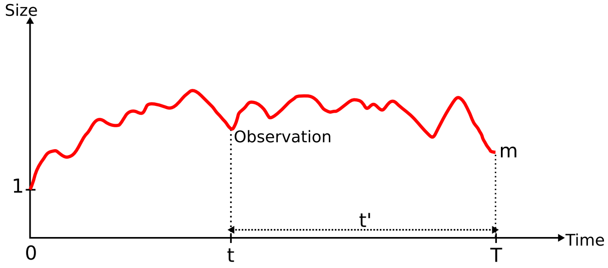
In this section we discuss the simple case of the Galton-Watson process [23, 27]. Our goal is to explain in this well-known example the approach that we will use in a more general context in Sects. 3–5.

2.1 Discrete Time

We first consider the discrete time case. The population is fully characterized by its size N_t at time t. At every time step, each individual is replaced by k offspring with probability p_k (in particular, p_0 is the probability that an individual leaves no descendance). Let us define the extinction probability $Q_e(t)$ as the probability that $N_t = 0$, given that we start with a single initial individual at time 0. During the first time step, the initial individual may branch into k offspring: thus, extinction of the initial lineage after $t + 1$ time steps is equivalent to the extinction of the lineages of its k offspring after t time steps. Since in this model the descendants of these offspring are independent, $Q_e(t)$ satisfies the following exact recursion:

$$Q_e(t+1) = F\left(Q_e(t)\right) \tag{2.1}$$

Fig. 2 Size N_t of the population at an intermediate time t knowing that the final size is $N_T = m$ at time $T = t + t'$



where

$$F(Q) = \sum_{k=0}^{\infty} p_k Q^k \tag{2.2}$$

with the initial condition $Q_e(0) = 0$. For large t, $Q_e(t)$ converges to a limit Q_e^* and $1 - Q_e^*$ is the probability of eternal survival of the population. This limit Q_e^* is the attractive fixed point of the map F, satisfying $Q_e^* = F(Q_e^*)$. The function $F(Q)$ is convex and satisfies $F(1) = 1$ and $F(0) = p_0 > 0$. The average number of offspring of an individual is given by $F'(1) = \sum_k p_k k = \overline{k}$. If $\overline{k} < 1$ the stable fixed point is $Q_e^* = 1$ and the population dies with probability 1. On the other hand when $\overline{k} > 1$, there is a non-zero survival probability $1 - Q_e^*$ in the $t \to \infty$ limit.

If one starts at $t = 0$ with a single individual, the distribution of the population size N_t can be characterized by its generating function

$$G_1(t; \mu) = \langle e^{-\mu N_t} \rangle. \tag{2.3}$$

The size N_t is the sum of the contributions of all the offspring at $t = 1$. The independence of the lineages of the k offspring implies that G_1 evolves according to the same equation (2.1) as $Q_e(t)$:

$$G_1(t + 1; \mu) = F\left(G_1(t; \mu)\right), \tag{2.4}$$

the only difference being in the initial condition $G_1(0; \mu) = e^{-\mu}$. Note that the extinction probability is given by the particular case $Q_e(t) = G_1(t; \infty)$.

For large T and a given finite size m, the events $N_T = m$ are more and more rare since the population size either vanishes or grows exponentially. For such events, however, one may be interested in the sizes N_t, $0 \le t \le T$, conditioned on the fact that $N_T = m$ (as in Fig. 2). To do so, let us divide the time interval $[0, T]$ into two intervals of lengths t and $t' = T - t$ and consider the two-time generating functions $G_2(t, t'; \mu, \nu)$ defined as:

$$G_2(t, t'; \mu, \nu) = \langle e^{-\mu N_t - \nu N_{t+t'}} \rangle. \tag{2.5}$$

As for Q_e and G_1, one can easily show, by considering the first time step, that $G_2(t, t'; \mu, \nu)$ as a function of t also evolves according to (2.1):

$$G_2(t + 1, t'; \mu, \nu) = F(G_2(t, t'; \mu, \nu)), \tag{2.6}$$

with the initial condition

$$G_2(0, t'; \mu, \nu) = e^{-\mu} G_1(t'; \nu). \tag{2.7}$$

If one expands $G_2(t,t';\mu,\nu)$ in powers of $e^{-\nu}$:

$$G_2(t,t';\mu,\nu) = \sum_{m=0}^{\infty} e^{-\nu m} R_m(t,t';\mu) \tag{2.8}$$

the coefficients $R_m(t,t';\mu)$ are related to the generating functions of N_t conditioned on the survival of m individuals at time T (i.e. $N_T = m$):

$$R_m(t,t';\mu) = P_m(t+t')\langle e^{-\mu N_t} | \text{size } N_{t+t'} = m\rangle \tag{2.9}$$

where $P_m(t+t')$ is the probability that $N_{t+t'} = m$. As in the special case $\mu = 0$

$$G_2(t,t';0,\nu) = G_1(t+t';\nu) = \sum_{m=0}^{\infty} e^{-\nu m} P_m(t+t') \tag{2.10}$$

one can extract the probability $P_m(t+t')$ from the expansion (2.10) of $G_1(t+t',\nu)$ in order to normalize $R_m(t,t';\mu)$ in (2.9) and get the conditional probability $\langle e^{-\mu N_t} | \text{size } m \text{ at } t+t'\rangle$.

We are now going to see that $\langle e^{-\mu N_t} | \text{size } m \text{ at } t+t' \rangle$ has a limit as t and $t' \to \infty$ and to show how to compute the distribution of N_t in this quasi-stationary regime. Since G_1 and G_2 evolve according to (2.4, 2.6), their long time behaviours can be extracted from the properties of F near the fixed point Q_e^*. Let u_t be a sequence obtained by iterating a map F

$$u_{t+1} = F(u_t) \tag{2.11}$$

with an initial condition u_0. If it converges exponentially to a fixed point Q_e^*, one can write:

$$u_t \simeq Q_e^* + \Lambda^t \mathcal{A}(u_0) + o(\Lambda^t), \quad \text{with } \Lambda = F'(Q_e^*) \tag{2.12}$$

where the amplitude $\mathcal{A}(u_0)$ is defined by:

$$\mathcal{A}(u_0) = \lim_{t\to\infty} \frac{u_t - Q_e^*}{\Lambda^t}. \tag{2.13}$$

Using (2.12) for G_2 and G_1 together with the initial condition (2.7) for large t and t' gives:

$$\begin{aligned} G_2(t,t';\mu,\nu) &= Q_e^* + \Lambda^t \mathcal{A}(e^{-\mu} G_1(t';\nu)) + o(\Lambda^t) \\ &= Q_e^* + \Lambda^t \mathcal{A}\big(e^{-\mu} Q_e^* + e^{-\mu} \Lambda^{t'} \mathcal{A}(e^{-\nu}) + \dots\big) + \dots \\ &= Q_e^* + \Lambda^t \mathcal{A}\left(e^{-\mu} Q_e^*\right) + \Lambda^{t+t'} \mathcal{A}'\left(e^{-\mu} Q_e^*\right) e^{-\mu} \mathcal{A}(e^{-\nu}) + \dots . \end{aligned} \tag{2.14}$$

If one expands $\mathcal{A}(e^{-\nu}) = \sum_m A_m e^{-\nu m}$, one gets from (2.8, 2.10, 2.14) at leading order in $\Lambda^{t+t'}$ for $m \geq 1$:

$$P_m(t+t') \simeq \Lambda^{t+t'} \mathcal{A}'(Q_e^*) A_m = \Lambda^{t+t'} A_m, \tag{2.15a}$$

$$R_m(t,t';\mu) \simeq \Lambda^{t+t'} \mathcal{A}'(e^{-\mu} Q_e^*) e^{-\mu} A_m. \tag{2.15b}$$

From these equations and (2.9), it is easy to see that for $t,t' \to \infty$:

$$\langle e^{-\mu N_t} | \text{size } N_{t+t'} = m \rangle \xrightarrow{t,t'\to\infty} \langle e^{-\mu N}\rangle_{\text{qs}}$$

with

$$\langle e^{-\mu N}\rangle_{\rm qs} = \mathcal{A}'(e^{-\mu} Q_e^*)e^{-\mu}. \tag{2.16}$$

This shows that the knowledge of the amplitude $\mathcal{A}$ defined in (2.13) determines the whole distribution of the population size in the quasi-stationary regime [28].

In general there is not a closed expression of $\mathcal{A}$ for a given map F. One can however relate the expansions of $\mathcal{A}$ and of F in the neighbourhood of the fixed point Q_e^*. The definition (2.13) implies that $\mathcal{A}$ satisfies:

$$\begin{aligned} \Lambda \mathcal{A}(Q) &= \mathcal{A}(F(Q)), \\ \mathcal{A}(Q_e^*) &= 0, \quad \mathcal{A}'(Q_e^*) = 1. \end{aligned} \tag{2.17}$$

By taking successive derivatives of (2.17) at $x = Q_e^*$, one can relate the Taylor expansions of $\mathcal{A}$ and F

$$\mathcal{A}(Q_e^* + \epsilon) = \epsilon + \frac{\epsilon^2}{2!}\mathcal{A}^{(2)} + \frac{\epsilon^3}{3!}\mathcal{A}^{(3)} + \dots$$

$$F(Q_e^* + \epsilon) = Q_e^* + \Lambda\epsilon + \frac{\epsilon^2}{2!}F^{(2)} + \frac{\epsilon^3}{3!}F^{(3)} + \dots$$

and get:

$$\mathcal{A}^{(2)} = \frac{F^{(2)}}{\Lambda - \Lambda^2} \tag{2.18a}$$

$$\mathcal{A}^{(3)} = \frac{F^{(3)}}{\Lambda(1-\Lambda)(1+\Lambda)} + \frac{3(F^{(2)})^2}{\Lambda(1+\Lambda)(1-\Lambda)^2}, \quad \text{etc.} \tag{2.18b}$$

Then, the moments of N_t in the quasi-stationary regime can be obtained by taking successive derivatives of (2.16) at $\mu = 0$:

$$\langle N\rangle_{\rm qs} = 1 + Q_e^*\mathcal{A}^{(2)} \tag{2.19a}$$

$$\langle N^2\rangle_{\rm qs} = 1 + 3Q_e^*\mathcal{A}^{(2)} + (Q_e^*)^2\mathcal{A}^{(3)}, \quad \text{etc.} \tag{2.19b}$$

As one approaches the transition point (i.e. $\overline{k} \to 1$ and $Q_e^* \to 1$), expressions (2.18, 2.19) diverge since $\Lambda \to 1$ and the moments scale as:

$$\langle N\rangle_{\rm qs} \simeq \mathcal{A}^{(2)} \simeq \frac{F^{(2)}}{1-\Lambda} \tag{2.20}$$

$$\frac{\langle N^2\rangle_{\rm qs}}{\langle N\rangle_{\rm qs}^2} \simeq \frac{\mathcal{A}^{(3)}}{(\mathcal{A}^{(2)})^2} \simeq \frac{3\Lambda}{(1+\Lambda)} + \frac{(1-\Lambda)F^{(3)}}{(1+\Lambda)(F^{(2)})^2} \tag{2.21}$$

and so on for higher moments $\langle N^p\rangle_{\rm qs}/\langle N\rangle_{\rm qs}^p$, $p > 2$. When the first moments of the number of offspring are finite (for example when $\overline{k^3}, \overline{k^4} < \infty$), the ratios such as $(1-\Lambda)F^{(3)}/((1+\Lambda)(F^{(2)})^2)$ go to zero as $\Lambda \to 1$ and one gets:

$$\langle N\rangle_{\rm qs} \simeq \frac{F^{(2)}(Q_e^*)}{1 - F'(Q_e^*)} \tag{2.22a}$$

$$\frac{\langle N^2\rangle_{\text{qs}}}{\langle N\rangle_{\text{qs}}^2} \to \frac{3}{2} \tag{2.22b}$$

$$\frac{\langle N^3\rangle_{\text{qs}}}{\langle N\rangle_{\text{qs}}^3} \to 3, \quad \text{etc.} \tag{2.22c}$$

One can notice that these expressions are valid on both sides of the transition, i.e. for $\overline{k} \to 1^+$ and $\overline{k} \to 1^-$ but do not depend otherwise on the p_k's. This agrees with the fact (see [23] and Sect. 2.3 below) that the distribution of N_t in the quasi-stationary regime becomes universal near the transition $\overline{k} = 1$.

2.2 Continuous Time Version

We now consider the continuous time version of the previous birth-death process: at any instant t, the population is fully characterized by its size N_t. During every infinitesimal time dt, each individual either dies with probability $\beta_0 dt$ or branches into k offspring with probability $\beta_k dt$. If one divides the time interval $[0, t+dt]$ into the first interval dt and the remaining interval $[dt, t+dt]$, the independence of the evolution of the offspring implies that the extinction probability $Q_e(t)$ satisfies as in (2.1):

$$Q_e(t+dt) = \left(1 - \sum_{k\geq 0} \beta_k dt\right) Q_e(t) + \sum_{k\geq 0} \beta_k dt\, Q_e(t)^k \tag{2.23}$$

which gives in the limit $dt \to 0$:

$$\partial_t Q_e(t) = F(Q_e(t)), \quad F(Q) = \sum_{k\geq 0} \beta_k (Q^k - Q). \tag{2.24}$$

This is the continuous time version of (2.1). In the long time limit, $Q_e(t)$ converges to the stable fixed point Q_e^* of F:

$$F(Q_e^*) = 0, \quad F'(Q_e^*) = \lambda < 0. \tag{2.25}$$

The transition occurs when $Q_e^* \to 1$ and $\lambda \to 0$. It corresponds to a growth rate

$$\alpha = \sum_{k\geq 0} \beta_k (k-1) \tag{2.26}$$

which vanishes. If $\alpha < 0$, $Q_e^* = 1$ whereas for $\alpha > 0$, one has $Q_e^* < 1$.

The generating functions G_1 and G_2 can be introduced as in (2.3, 2.5) and the discussion of their long time behaviours is the same as in (2.14). For a flow $u(t)$ such that $\partial_t u(t) = F(u(t))$ and $u(0) = u_0$, one can define an amplitude $\mathcal{A}$ as in (2.12) by:

$$\mathcal{A}(u_0) = \lim_{t\to\infty} \frac{u(t) - Q_e^*}{e^{\lambda t}}. \tag{2.27}$$

The generating functions of the size N_t in the quasi-stationary regime are still given by (2.16) but the definition of $\mathcal{A}$ for continuous time leads to the following continuous time version of (2.17):

$$\lambda \mathcal{A}(Q) = \mathcal{A}'(Q) F(Q). \tag{2.28}$$

As in (2.18), the expansions of $\mathcal{A}$ and F near the fixed point Q_e^* can be related through:

$$\mathcal{A}^{(2)} = -\frac{F^{(2)}}{\lambda}, \tag{2.29a}$$

$$\mathcal{A}^{(3)} = \frac{3}{2}\left(\frac{F^{(2)}}{\lambda}\right)^2 - \frac{F^{(3)}}{2\lambda}. \tag{2.29b}$$

The average size is given by $\langle N\rangle_{\text{qs}} = -F^{(2)}/\lambda$ and diverges as one approaches the transition $\alpha \to 0$. For models with finite moments $\overline{k^n}$, the first moments $\langle N^n\rangle_{\text{qs}}$ obtained from the expansion (2.29) of $\mathcal{A}$ go to the same values as in (2.22).

As (2.28) is a first order differential equation, one can fully determine $\mathcal{A}$ using $\mathcal{A}'(Q_e^*) = 1$ (which follows from definition (2.28)):

$$\mathcal{A}(Q) = (Q - Q_e^*)\exp\left[\int_{Q_e^*}^{Q}\left(\frac{\lambda}{F(y)} - \frac{1}{y - Q_e^*}\right)dy\right]. \tag{2.30}$$

2.3 Universality of the Stationary Regime

Near the transition, Q_e^* is close to 1, λ is small and for Q close to Q_e^*, one can approximate $F(Q)$ by the first two terms of its quadratic expansion near Q_e^* (when $\overline{k^2} < \infty$):

$$F(Q) \simeq \lambda(Q - Q_e^*) + B(Q - Q_e^*)^2. \tag{2.31}$$

Then (2.30) becomes near Q_e^*:

$$\mathcal{A}(Q) \simeq \frac{\lambda(Q - Q_e^*)}{\lambda + B(Q - Q_e^*)} \tag{2.32}$$

and from (2.16), one gets that for μ small:

$$\langle e^{-\mu N}\rangle_{\text{qs}} \simeq \frac{1}{(1 - \frac{B}{\lambda}\mu)^2}. \tag{2.33}$$

Close to the transition, $B \simeq F''(1)/2 \simeq (\overline{k^2} - \overline{k})/2$ and this shows that in the quasi-stationary regime:

$$\langle N\rangle_{\text{qs}} \simeq -\frac{\overline{k^2} - \overline{k}}{\lambda} \tag{2.34}$$

and the distribution of the ratio $x = N/\langle N\rangle_{\text{qs}}$ becomes universal [23]:

$$P(x) \simeq 4xe^{-2x} \tag{2.35}$$

with the ratios of the first moments given by (2.22).

The case of slowly decreasing branching rates β_k for which the second moment $\overline{k^2}$ is infinite leads to other universal distributions near the transition, as shown in [40] and in Appendix A.

3 Quasi-Stationary Regime for a Branching Random Walk in Presence of an Absorbing Wall

3.1 Evolution of the Extinction Probability

We now consider a population for which each individual is characterized by its position x relative to an absorbing wall. Each individual diffuses and branches into k offspring with probability $\beta_k dt$ during dt (for simplicity we set $\beta_0 = 0$, *i.e.* there is no spontaneous death; also there is no dependence on β_1 as it has no effect on the population). The absorbing wall moves at constant velocity v and every individual crossing the wall disappears instantaneously (see Fig. 1). In absence of the wall, the population grows exponentially and occupies a region which spreads in space linearly at a constant velocity v_c [5, 31]. The presence of the wall introduces a competition between the exponential growth and the absorption by the wall. For $v > v_c$, the wall moves faster than the population spreads [12] and the population dies with probability 1.

If one starts at time $t = 0$ with a single individual at distance x from the moving wall, the extinction probability $Q_e(x,t)$ (*i.e.* the probability that all the descendants have been absorbed by the wall during the interval $[0,t]$) evolves according to:

$$Q_e(x+vdt, t+dt) = \int \frac{e^{-\eta^2/4}}{\sqrt{4\pi}} Q_e(x+\eta\sqrt{dt}, t) d\eta + \sum_{k=2}^{\infty} \beta_k \left(Q_e(x,t)^k - Q_e(x,t) \right) dt. \tag{3.1}$$

To derive this equation, we divide, as in Sect. 2, the time interval $[0, t+dt]$ into a first time step dt and a second interval $[dt, t+dt]$. During the first infinitesimal time step dt, the individual diffuses (its position is shifted by an amount $\eta\sqrt{dt}$ where η is a Gaussian random variable such that $\langle\eta\rangle = 0$ and $\langle\eta^2\rangle = 2$) and may branch into $k \geq 2$ individuals with rate β_k. If this branching event happens, then, during the second time interval $[dt, t+dt]$, all the k offspring present at time dt have independent histories.

In the limit $dt \to 0$, this becomes:

$$\partial_t Q_e = \mathcal{F}(Q_e) \tag{3.2}$$

where the functional $\mathcal{F}$ is defined as:

$$\mathcal{F}(Q) = \partial_x^2 Q(x) - v\partial_x Q(x) + g(Q(x)) \tag{3.3}$$

$$g(q) = \sum_{k=2}^{\infty} \beta_k (q^k - q). \tag{3.4}$$

Q_e evolves according to a traveling wave equation [43]. Here however the evolution is limited to a semi-infinite line with the boundary condition $Q_e(0,t) = 1$ on the wall. The non-linear function $g(q)$ contains all the information on the branching rates β_k of the walk. In the long time limit $t \to \infty$, the extinction probability $Q_e(x,t)$ converges to the stable solution Q_e^* of $\mathcal{F}(Q_e^*) = 0$ and therefore satisfies:

$$\partial_x^2 Q_e^* - v\partial_x Q_e^* + g\left(Q_e^*(x)\right) = 0. \tag{3.5}$$

For $v > v_c$, the wall moves faster than the spreading velocity v_c of the population, so that all the population gets absorbed and for large t, $Q_e(x,t)$ has the shape of a front moving at velocity $v - v_c$. Therefore $Q_e(x,t)$ converges to the unique stable fixed point $Q_e^* = 1$ (the population becomes extinct with probability 1 whatever the initial distance x to the wall is).

For $v < v_c$, there exists a non trivial fixed point Q_e^* with $Q_e^*(0) = 1$ and $Q_e^*(x) \to 0$ for $x \to \infty$ and $Q_e(x,t) \to Q_e^*(x)$ as $t \to \infty$, with relaxation times τ_n:

$$Q_e(x,t) \simeq Q_e^* + A_1\phi_1(x)e^{-t/\tau_1} + \dots. \tag{3.6}$$

These relaxation times τ_n are related through $\lambda_n = -1/\tau_n$ to the eigenvalues λ_n of the linear operator $\mathcal{L}$ obtained by linearizing the functional $\mathcal{F}(Q_e^* + \epsilon\phi) = \epsilon\mathcal{L}[\phi] + o(\epsilon)$ around the fixed point Q_e^*. The eigenvectors $\phi_n(x)$ and the eigenvalues λ_n satisfy:

$$\mathcal{L}[\phi_n] = \lambda_n\phi_n \tag{3.7}$$

with the linear operator $\mathcal{L}$ given by:

$$\mathcal{L}[\phi] = \partial_x^2\phi - v\partial_x\phi + g'\left(Q_e^*(x)\right)\phi. \tag{3.8}$$

Defining $\psi(x) = \phi(x)e^{-vx/2}$ maps (3.8) to a Schrödinger equation

$$-\partial_x^2\psi(x) + V(x)\psi(x) = (-\lambda)\psi(x) \tag{3.9}$$

with a potential V given by:

$$V(x) = \frac{v^2}{4} - \sum_k \beta_k(kQ_e^*(x)^{k-1} - 1) \tag{3.10}$$

(note that for $v > v_c$, one has $Q_e^*(x) = 1$, so the potential V is constant and the spectrum is continuous. For $v > v_c$, $Q_e(x,t)$ converges to $Q_e^*(x)$ in a non uniform way, so that there are always some x for which $Q_e(x,t) - Q_e^*(x)$ is not small and the long time behaviour of $Q_e(x,t)$ cannot be understood by linearizing around the fixed point $Q_e^*(x) = 1$).

As v approaches v_c from below, the region where $Q_e^*(x)$ is close to 1 increases (and diverges when $v = v_c$). As a consequence, as $v \to v_c$, the potential well $V(x)$ in (3.10) has a finite depth but an increasing length and this gives rise to a discrete spectrum for small eigenvalues. We will calculate the eigenvalues λ_n of $\mathcal{L}$ in this limit $v \to v_c$ in Sect. 5.1 and Appendix B.

In the rest of this paper we will only consider the case $v < v_c$ when the smallest eigenvalues of $\mathcal{L}$ form a discrete spectrum (only the construction of Sect. 4.1 will be also valid for $v > v_c$ as one does not need there the slowest eigenvalue to be isolated).

In contrast to (2.3), the population is now not only characterized by its size N_t but also by the positions of all the individuals. One can however follow the same approach as in Sect. 2. Let us introduce the generating function G_1:

$$G_1(x,t;f) = \left\langle \prod_{i=1}^{N_t} e^{-f(x_i^{(t)})} \right\rangle \tag{3.11}$$

where f is now a positive test function of x and plays the role of μ in (2.3), N_t is the size of the population at time t generated by an initial individual at x and the $x_i^{(t)}$ are the positions relatively to the wall of all its offspring at t. The independence of the lineages implies that

G_1 for the initial individual is the product of the G_1's of its offspring after the first time step dt. Therefore, the evolution of $G_1(x,t;f)$ can be derived as in (3.1) for $Q_e(x,t)$ and one gets:

$$\partial_t G_1(x,t;f) = \mathcal{F}\big(G_1(x,t;f)\big). \tag{3.12}$$

Absorption by the wall implies $G_1(0,t;f)=1$ for all $t>0$ and, for a single initial individual at x when $t=0$, the initial condition for G_1 is:

$$G_1(x,0;f) = e^{-f(x)}. \tag{3.13}$$

3.2 Conditioning on Survival

As in Sect. 2, one can then try to calculate the size N_t of the population or the density $\rho(x)$ at time t conditioned on the survival of m individuals at later time $T=t+t'$ (see Fig. 2). To do so we introduce the two-time generating function at $G_2(x,t,t';\mu,\nu)$ defined as in (2.5):

$$G_2(x,t,t';f,\nu) = \left\langle \exp\left(-\sum_{i=1}^{N_t} f(x_i^{(t)}) - \nu N_{t+t'}\right)\right\rangle \tag{3.14}$$

given that at $t=0$ there is a single individual at distance x from the wall. Once more, one can show that $G_2(x,t,t';f,\nu)$ (as a function of x and t only) evolves as $G_1(x,t;f)$, i.e. according to (3.12) with the following initial condition:

$$G_2(x,0,t';f,\nu) = e^{-f(x)} G_1(x,t';\nu). \tag{3.15}$$

As in (2.8–2.10), one can expand G_2 in powers of $e^{-\nu}$:

$$G_2(x,t,t';f,\nu) = \sum_{m=0}^{\infty} e^{-\nu m} R_m(x,t,t';f) \tag{3.16}$$

$$G_2(x,t,t';0,\nu) = G_1(x,t+t';\nu) = \sum_{m=0}^{\infty} e^{-\nu m} P_m(x,t+t') \tag{3.17}$$

where $P_m(x,t+t')$ is the probability that the initial individual located at x has exactly m living descendants at $t+t'$ and the generating function of f conditioned on observing a size m at $t+t'$ is given by:

$$\left\langle \exp\left(-\sum_{i=1}^{N_t} f(x_i^{(t)})\right)\middle| \text{size } m \text{ at } t+t' \right\rangle = \widetilde{R}_m(x,t,t';f) \tag{3.18}$$

where $\widetilde{R}_m(x,t,t';f)$ is defined as:

$$\widetilde{R}_m(x,t,t';f) = \frac{R_m(x,t,t';f)}{P_m(x,t+t')}. \tag{3.19}$$

This shows that all the information about the regime $0<t<T$, conditioned on having the final size $N_T=m$, is contained in the functions $\widetilde{R}_m(x,t,t';f)$. Equations (3.15–3.17) are the analogues of (2.7, 2.8, 2.10) in Sect. 2.

Inserting the expansions (3.16, 3.17) into (3.12) leads to differential equations for the R_m's and the P_m's. At lowest order, R_0 and P_0 satisfy (3.12) and, at first order, R_1 and P_1 are related to R_0 and P_0 through:

$$\partial_t R_1 = \partial_x^2 R_1 - v\partial_x R_1 + \sum_{k=2}^{\infty} \beta_k (k R_0^{k-1} - 1) R_1 \tag{3.20a}$$

$$\partial_t P_1 = \partial_x^2 P_1 - v\partial_x P_1 + \sum_{k=2}^{\infty} \beta_k (k P_0^{k-1} - 1) P_1 \tag{3.20b}$$

$$R_1(x, 0, t'; f) = e^{-f(x)} P_1(x, t'), \quad P_1(x, 0) = 1. \tag{3.20c}$$

By expanding further G_2 in powers of $e^{-\nu}$, one could obtain in the same way the evolution equations of the R_m's and P_m's and from (3.15) their initial values.

3.3 The Quasi-Stationary Regime

In this subsection, we show how one can generalize the expression (2.16) of Sect. 2 relating the properties of the quasi-stationary regime to those of the functional $\mathcal{F}$ defined in (3.3) near the fixed point Q_e^* for $v < v_c$. An alternative (easier) approach to calculate these properties will be described and used in Sects. 4 and 5.

The generating functions G_1 and G_2 evolve according to (3.12) and, as soon as the test function f is positive, G_1 and G_2 converge to the non-trivial fixed point Q_e^* of $\mathcal{F}$ when $v < v_c$. In order to analyze the long time behaviour of G_1 and G_2, one can expand $\mathcal{F}$ around Q_e^*:

$$\mathcal{F}(Q_e^* + \epsilon\phi) = \mathcal{F}(Q_e^*) + \epsilon\mathcal{L}[\phi] + \frac{\epsilon^2}{2!}\mathcal{F}^{(2)}[\phi, \phi] + \dots \tag{3.21}$$

where the n-linear symmetrical functionals $\mathcal{F}^{(n)}$ are defined for any arbitrary set of n functions ψ_i as:

$$\mathcal{F}^{(n)}[\psi_1, \dots, \psi_n] = \frac{d^n \mathcal{F}(Q_e^* + s_1\psi_1 + \dots + s_n\psi_n)}{ds_1 \dots ds_n}\bigg|_{\mathbf{s}=0}. \tag{3.22}$$

As discussed after (3.10), the linear operator $\mathcal{L}$ does not always have a discrete spectrum (in particular for $v > v_c$). We will assume here that $v < v_c$ and that at least the smallest eigenvalue λ_1 is isolated. If the first eigenvalue is not isolated, all the construction below breaks down and the quasi-stationary state might not exist [12].

As Q_e, G_1 and G_2 all evolve according to (3.12), one can study the dynamical system:

$$\begin{cases} \partial_t u(x,t) = \mathcal{F}(u(x,t)), \\ u(x,0) = u_0(x). \end{cases} \tag{3.23}$$

When $u(x,t) \to Q_e^*(x)$ as $t \to \infty$ and if $\mathcal{L}$ has a discrete spectrum for small eigenvalues, the convergence of $u(x,t)$ is exponential. Then one can introduce as in (2.13) the amplitude $\mathcal{A}$ of the first eigenvector ϕ_1 of $\mathcal{L}$ (with the largest relaxation time τ_1):

$$u(x,t) = Q_e^* + \phi_1(x) e^{-t/\tau_1} \mathcal{A}(u_0(\cdot)) + \dots \tag{3.24}$$

but now the argument of the amplitude $\mathcal{A}$ is a function (the initial condition $u_0(x) = u(x,0)$).

From the initial conditions (3.13, 3.15) of G_1 and G_2, the generating functions $G_1(x,t;f)$ and $G_2(x,t,t';f,\nu)$ thus have the following behaviours as $t\to\infty$:

$$G_1(x,t;f) = Q_e^*(x) + \phi_1(x)e^{-t/\tau_1}\mathcal{A}(e^{-f}) + o(e^{-t/\tau_1})$$

$$G_2(x,t,t';f,\nu) = Q_e^*(x) + \phi_1(x)e^{-t/\tau_1}\mathcal{A}\left(e^{-f}G_1(\cdot,t';\nu)\right) + o(e^{-t/\tau_1})$$

Thus for large t and t', at leading orders, $G_2(x,t,t';f,\nu)$ becomes:

$$\begin{aligned} G_2(x,t,t';f,\nu) &\simeq Q_e^*(x) + \phi_1(x)e^{-t/\tau_1}\mathcal{A}\big(e^{-f}\big(Q_e^* + e^{-t'/\tau_1}\phi_1\mathcal{A}(e^{-\nu})\big)\big) \\ &\simeq Q_e^*(x) + \phi_1(x)e^{-t/\tau_1}\mathcal{A}\left(e^{-f}Q_e^*\right) \\ &\quad + \phi_1(x)e^{-(t+t')/\tau_1}\frac{d}{ds}\mathcal{A}\left(e^{-f}Q_e^* + se^{-f}\phi_1\right)\Big|_{s=0}\mathcal{A}(e^{-\nu}) + \dots. \end{aligned} \tag{3.25}$$

As $G_2(x,t,t';0,\nu) = G_1(x,t+t';\nu)$, one can proceed as in (2.15) by writing $\mathcal{A}(e^{-\nu}) = \sum_m A_m e^{-\nu m}$ and get for R_m and P_m defined in (3.16, 3.17) for $m \geq 1$:

$$R_m(x,t,t';f) \simeq A_m\phi_1(x)e^{-(t+t')/\tau_1}\frac{d}{ds}\mathcal{A}\left(e^{-f}Q_e^* + se^{-f}\phi_1\right)\Big|_{s=0} \tag{3.26a}$$

$$P_m(x,t+t') \simeq A_m\phi_1(x)e^{-(t+t')/\tau_1}. \tag{3.26b}$$

Thus, the generating function of f given by (3.16), when $t,t'\to\infty$, has a finite limit:

$$\lim_{t,t'\to\infty}\left\langle \exp\left(-\sum_{i=1}^{N_t} f(x_i^{(t)})\right)\middle|\text{size } N_{t+t'} = m\right\rangle = \left\langle \exp\left(-\sum_{i=1}^{N_t} f(x_i^{(t)})\right)\right\rangle_{\text{qs}} \tag{3.27}$$

given by:

$$\left\langle \exp\left(-\sum_{i=1}^{N_t} f(x_i^{(t)})\right)\right\rangle_{\text{qs}} = \frac{d}{ds}\mathcal{A}\left(e^{-f}Q_e^* + se^{-f}\phi_1\right)\Big|_{s=0} \tag{3.28}$$

(note that as in (3.24) the argument of $\mathcal{A}$ is a function). We see that, as in (2.16), the properties of the quasi-stationary state are determined in (3.28) by the amplitude $\mathcal{A}$ (expression (3.28) does not depend anymore on t, t', nor on the value of m used to condition the population size at $t+t'$, nor on the position x of the initial individual).

From the definition (3.23, 3.24), one can see that the amplitude $\mathcal{A}$ satisfies:

$$\lambda_1\mathcal{A}(u) = \frac{d}{d\tau}\mathcal{A}(u+\tau\mathcal{F}(u))\Big|_{\tau=0}. \tag{3.29}$$

As in Sect. 2, this equation can be used to relate $\mathcal{A}$ and $\mathcal{F}$. Note that equations similar to (3.29) appear in other contexts, in particular in the renormalization group theory where $\mathcal{A}$ is called a non-linear scaling field [44]. As in (3.21), one can expand $\mathcal{A}$ around Q_e^*:

$$\mathcal{A}(Q_e^*+\epsilon\phi) = \mathcal{A}(Q_e^*) + \epsilon\mathcal{A}^{(1)}[\phi] + \frac{\epsilon^2}{2!}\mathcal{A}^{(2)}[\phi,\phi] + \dots. \tag{3.30}$$

Successive derivatives of (3.29) as defined in (3.22) in the directions of the eigenvectors ϕ_n of $\mathcal{L}$ allow one to relate the expansions (3.30, 3.21) of $\mathcal{A}$ and $\mathcal{F}$:

$$\mathcal{A}^{(1)}[\phi_i] = \delta_{i,1} \tag{3.31a}$$

$$\mathcal{A}^{(2)}[\phi_i, \phi_j] = \frac{1}{\lambda_1 - \lambda_i - \lambda_j} \mathcal{A}^{(1)}[\mathcal{F}^{(2)}[\phi_i, \phi_j]] \tag{3.31b}$$

$$\begin{aligned} &\mathcal{A}^{(3)}[\phi_i, \phi_j, \phi_k] \\ &\quad = \frac{1}{\lambda_1 - \lambda_i - \lambda_j - \lambda_k} \big(\mathcal{A}^{(1)}\big[\mathcal{F}^{(3)}[\phi_i, \phi_j, \phi_k]\big] + \mathcal{A}^{(2)}\big[\mathcal{F}^{(2)}[\phi_i, \phi_j], \phi_k\big] \\ &\qquad + \mathcal{A}^{(2)}\big[\mathcal{F}^{(2)}[\phi_i, \phi_k], \phi_j\big] + \mathcal{A}^{(2)}\big[\mathcal{F}^{(2)}[\phi_j, \phi_k], \phi_i\big]\big). \end{aligned} \tag{3.31c}$$

Then, from (3.28), one can obtain the properties of the quasi-stationary regime. For example, the moments of the number of individuals in the quasi-stationary state are obtained by taking $f = \mu$ constant and taking successive derivatives at $\mu = 0$:

$$\begin{aligned} \langle N \rangle_{\text{qs}} &= -\frac{d^2}{ds\,d\mu} \mathcal{A}(Q_e^* e^{-\mu} + s\phi_1 e^{-\mu})\Big|_{\mu=s=0} \\ &= \mathcal{A}^{(1)}[\phi_1] + \mathcal{A}^{(2)}[\phi_1, Q_e^*] = 1 + \mathcal{A}^{(2)}[\phi_1, Q_e^*] \end{aligned} \tag{3.32}$$

$$\begin{aligned} \langle N^2 \rangle_{\text{qs}} &= \frac{d^3}{ds\,d\mu^2} \mathcal{A}(Q_e^* e^{-\mu} + s\phi_1 e^{-\mu})\Big|_{\mu=s=0} \\ &= \mathcal{A}^{(1)}[\phi_1] + 2\mathcal{A}^{(2)}[\phi_1, Q_e^*] + \mathcal{A}^{(3)}[\phi_1, Q_e^*, Q_e^*] \end{aligned} \tag{3.33}$$

These expressions are exact and valid as long as the functional $\mathcal{F}$ has a non-trivial fixed point Q_e^* and an isolated largest relaxation time τ_1. In order to go further, one needs to know more precisely Q_e^* and ϕ_1. This will be done in Sect. 5 in the scaling regime $v \to v_c$.

4 A Modified Process to Describe Conditioned Histories

Before analyzing this scaling regime we are going to show that one can construct a modified process reproducing the history $0 \le t \le T$ of the branching random walk of Sect. 3 conditioned on a final size $N_T = 1$. This construction is valid both below and above the critical velocity v_c. The quasi-stationary state for $v < v_c$ discussed in Sect. 3.3 will appear as a particular case and its average profile will be calculated in Sect. 4.2.

4.1 Construction of the Modified Process

One way of describing branching random walks conditioned on the survival of a single individual is to distinguish the path of the surviving particle from the other ones. In the mathematical literature [21, 22], this particle is called the *spine* and has its dynamics modified in order to prevent it from dying. The rest of the population is generated from this special particle. We show in the present section that this construction is general for branching random walks conditioned on having exactly one survivor at finite time T and requires only the knowledge of the extinction probability $Q_e(x,t)$ and the probability $P_1(x,t)$ that there is exactly one survivor at t.

To condition the evolution of the population on the survival of one individual at time $t+t'$, (3.18, 3.19) show that one should consider the ratio $\widetilde{R}_1(x,t,t';f) = R_1(x,t,t';f)/P_1(x,t+t')$. From (3.12, 3.20a) satisfied by R_0 and R_1, one can show that $\widetilde{R}_0$ and $\widetilde{R}_1$ satisfy:

$$\partial_t \widetilde{R}_0 = \partial_x^2 \widetilde{R}_0 + \left(-v + 2\frac{\partial_x Q_e(x,t+t')}{Q_e(x,t+t')}\right)\partial_x \widetilde{R}_0 + \sum_{k=2}^{\infty} \beta_k Q_e(x,t+t')^{k-1}\left(\widetilde{R}_0^k - \widetilde{R}_0\right) \tag{4.1}$$

$$\partial_t \widetilde{R}_1 = \partial_x^2 \widetilde{R}_1 + \left(-v + 2\frac{\partial_x P_1(x,t+t')}{P_1(x,t+t')}\right)\partial_x \widetilde{R}_1 + \sum_{k=2}^{\infty} k\beta_k Q_e(x,t+t')^{k-1}\left(\widetilde{R}_0^{k-1}\widetilde{R}_1 - \widetilde{R}_1\right) \tag{4.2}$$

(where we have used the fact that $Q_e(x,t) = P_0(x,t)$). The initial conditions (3.20c) give $\widetilde{R}_0(x,0,t';f) = \widetilde{R}_1(x,0,t';f) = e^{-f(x)}$.

We are now going to show that the functions $\widetilde{R}_0$ and $\widetilde{R}_1$ can be interpreted as the generating functions at time t of a modified process defined on $[0,T]$ with $T = t+t'$. To do so, we consider the following process for $0 < t < T$ in the frame of the wall:

- the system consists of a single particle of type A_1 (the "spine" particle) and an arbitrary number of particles A_0;
- a particle of type A_0 diffuses at time t in the frame of the wall with a drift

$$v_0(x,t,T) = -v + 2\frac{\partial_x Q_e(x,T-t)}{Q_e(x,T-t)} \tag{4.3}$$

  and branches into k particles of type A_0 at rate $\beta_k^{(0)}(x,t,T) = \beta_k Q_e(x,T-t)^{k-1}$:

$$A_0 \xrightarrow{\beta_k^{(0)}(x,t,T)} kA_0 \tag{4.4}$$

- the particle of type A_1 diffuses in the frame of the wall with a drift

$$v_1(x,t,T) = -v + 2\frac{\partial_x P_1(x,T-t)}{P_1(x,T-t)} \tag{4.5}$$

  and branches into one particle of type A_1 and $k-1$ particles of type A_0 at rate $\beta_k^{(1)}(x,t,T) = k\beta_k Q_e(x,T-t)^{k-1}$:

$$A_1 \xrightarrow{\beta_k^{(1)}(x,t,T)} A_1 + (k-1)A_0. \tag{4.6}$$

One can notice that now the drifts and the mutation rates depend both on space and time through the functions $Q_e(x,t)$ and $P_1(x,t)$. Moreover, as $P_1(0,t) = 0$ and $\partial_x P_1(0,t) \neq 0$, the particle A_1 is never absorbed by the wall (whenever it approaches the wall, it is pushed away (4.5) from it by the drift $\partial_x P_1/P_1$ which diverges when $x \to 0$). The extinction probability $Q_e^{(0)}(x,t;t')$ of a particle A_0 is given by $Q_e^{(0)}(x,t;t') = Q_e(x,t)/Q_e(x,t+t')$ and one can verify that it satisfies (4.1): indeed it is equal to $\widetilde{R}_0(x,t,t';\infty)$. Thus, at $t'=0$, i.e. at the final time T, all particles A_0 have disappeared.

For this modified process, one can consider the new generating function $\widetilde{G}^{(0)}(x,t,T,f) = \langle \exp(-\sum_i f(x_i^{(t)}))\rangle$ (resp. $\widetilde{G}^{(1)}(x,t,T,f)$) where the summation at time t is over both particles A_0 and A_1, given that we start with a single initial individual of type A_0 at position x (resp. A_1) at $t=0$. In this modified process, the offspring are still independent as in (3.1) and the function $\widetilde{G}^{(0)}$ (resp. $\widetilde{G}^{(1)}$) is the product of the $\widetilde{G}^{(i)}$'s of the offspring of the initial individual of type A_0 (resp. A_1). As in (3.1), one can split the time interval $[0, T+dt]$ into two intervals $[0, dt]$ and $[dt, T+dt]$ and one gets:

$$
\begin{aligned}
&\widetilde{G}^{(0)}(x, t+dt, T+dt, f)\\
&\quad = \int \frac{e^{-\eta^2/4}}{\sqrt{4\pi}} \widetilde{G}^{(0)}(x+\eta\sqrt{dt}+v_0(x,0,T+dt)dt, t, T, f)d\eta\\
&\qquad + \sum_k \beta_k Q_e(x, T+dt)^{k-1}\left(\widetilde{G}^{(0)}(x,t,T,f)^k - \widetilde{G}^{(0)}(x,t,T,f)\right)
\end{aligned}
\tag{4.7}
$$

and, as dt is infinitesimal, $\widetilde{G}^{(0)}$ has the same evolution (4.1) as $\widetilde{R}_0$. Similarly, writing the evolution of $\widetilde{G}^{(1)}$ in the same way shows that it has the same evolution as $\widetilde{R}_1$. Since the $\widetilde{G}^{(0)}$ and $\widetilde{R}_0$ (resp. $\widetilde{G}^{(1)}$ and $\widetilde{R}_1$) have also the same initial conditions, one has $\widetilde{R}_0(x,t,t';f) = \widetilde{G}^{(0)}(x,t,T,f)$ and $\widetilde{R}_1(x,t,t';f) = \widetilde{G}^{(1)}(x,t,T,f)$ and the $\widetilde{R}_i$'s introduced in (3.19) can be interpreted as the generating functions of a modified process defined on $[0,T]$.

When $T\to\infty$ and $v<v_c$, the extinction probability $Q_e(x,T-t)$ converges to $Q_e^*(x)\neq 0$ and $P_1(x,T-t)$ decreases as in (3.26). Thus, as $T\to\infty$, the drifts and the branching rates of the modified process become independent of t:

$$
v_0(x,t,T) \to w_0(x) = -v + 2\frac{\partial_x Q_e^*(x)}{Q_e^*(x)}, \quad \beta_k^{(0)}(x,t,T) \to \beta_k Q_e^*(x)^{k-1} \tag{4.8a}
$$

$$
v_1(x,t,T) \to w_1(x) = -v + 2\frac{\partial_x \phi_1(x)}{\phi_1(x)}, \quad \beta_k^{(1)}(x,t,T) \to k\beta_k Q_e^*(x)^{k-1}. \tag{4.8b}
$$

From the expressions (4.8), one can understand the best strategy for the system to have a single survivor at $T=t+t'$. The branching rates decrease as x increases and vanish as $Q_e^*(x)$ goes to 0: no population can develop in the region where $Q_e^*(x)\simeq 0$ and it prevents the population from growing exponentially (which would not be compatible with a finite size at T). The particle A_1 cannot be absorbed because $v_1(x,t,T)\to\infty$ as $x\to 0$.

Remark 1 For the birth-death process of Sect. 2.2, one could similarly construct a modified process [21] with branching rates $\beta_k Q_e^{*k-1}$ for particles A_0 and $k\beta_k Q_e^{*k-1}$ for the *spine* A_1, to describe the quasi-stationary regime.

Remark 2 The construction of a modified process can be easily adapted to intermediate regimes $0<t<T$ conditioned on the survival of two (or more generally to p) survivors at T. The differential equations satisfied by R_2 and P_2 could in the same way be interpreted as the dynamics of a system of three types of particles with the following branching rates:

$$
\begin{aligned}
A_0 &\to kA_0, && \text{rate } \beta_k Q_e(x,T-t)^{k-1}\\
A_1 &\to A_1 + (k-1)A_0, && \text{rate } k\beta_k Q_e(x,T-t)^{k-1}\\
A_2 &\to A_2 + (k-1)A_0, && \text{rate } k\beta_k Q_e(x,T-t)^{k-1}\\
A_2 &\to A_1 + A_1, && \text{rate } \frac{k(k-1)}{2}\beta_k Q_e(x,T-t)^{k-2}P_1(x,T-t)
\end{aligned}
$$

where A_0 and A_1 have the same drifts as before and A_2 has an additional drift $2\partial_x P_2(x, T-t)/P_2(x, T-t) \simeq 2\partial_x \phi_1(x)/\phi_1(x)$ for $T \to \infty$. The exponential decay of P_1 to 0 with a relaxation time τ_1 shows that the reaction $A_2 \to A_1 + A_1$ occurs only for $t \simeq T$.

4.2 Average Profiles

The average density profile $\rho(X,t)$ of the population is defined such that $\rho(X,t)dX$ is the average number of individuals located in the interval $[X, X+dX]$ in the moving frame of the wall. It is easy to see that it satisfies:

$$\partial_t \rho = \partial_X^2 \rho + v\partial_X \rho + \sum_k \beta_k (k-1)\rho \tag{4.9}$$

with $\rho(0,t) = 0$ because of the absorption by the wall and $\rho(X,0) = \delta(X-x)$ where x is the position of the initial individual. If one introduces the growth rate $\alpha = \sum_k \beta_k(k-1)$, the solution is given by:

$$\rho(X,t) = \frac{1}{\sqrt{4\pi t}} e^{(\alpha - v^2/4)t} e^{-v(X-x)} \left[e^{-(X-x)^2/4t} - e^{-(X+x)^2/4t} \right]. \tag{4.10}$$

Similar expressions were obtained in [1] for a model of cell proliferation.

If $v > v_c = 2\alpha^{1/2}$, the average density decreases to zero and this corresponds to an almost sure extinction of the population. For $v < v_c$, the divergence corresponds to the exponential growth whenever the population survives.

We are now going to calculate the average profile $\rho_{\mathrm{qs}}(X)$ in the quasi-stationary regime using the modified process of Sect. 4.1. There are two contributions to this profile:

$$\rho_{\mathrm{qs}}(X) = \rho_{1,\mathrm{st}}(X) + \rho_{0,\mathrm{st}}(X) \tag{4.11}$$

where $\rho_{1,\mathrm{st}}(X)$ and $\rho_{0,\mathrm{st}}(X)$ are the stationary average density profiles of the particles A_1 and A_0 in the modified process. In the moving frame of the wall and in the stationary regime, particles of type A_1 (resp. A_0) have a drift $w_1(X) = -v + 2\partial_x\phi_1/\phi_1$ (resp. $w_0(X) = -v + 2\partial_x Q_e^*/Q_e^*$). The average density profiles ρ_0 and ρ_1 satisfy equations similar to (4.9) where we use now the drifts and the branching rates (4.8) of the modified process:

$$\begin{aligned} \partial_t \rho_0(X,t) = \partial_X^2 \rho_0(X,t) &- \partial_X \left(w_0(X) \rho_0(X,t) \right) \\ &+ \sum_k (k-1)\beta_k Q_e^*(X)^{k-1} \rho_0(X,t) \\ &+ \sum_k k(k-1)\beta_k Q_e^*(X)^{k-1} \rho_1(X,t) \end{aligned} \tag{4.12}$$

$$\partial_t \rho_1(X,t) = \partial_X^2 \rho_1(X,t) - \partial_X \left(w_1(X) \rho_1(X,t) \right). \tag{4.13}$$

There is no source term for ρ_1 since the number of particles A_1 is conserved. As there is initially a single particle A_1, the density ρ_1 is the distribution of its position at time t. The stationary profile $\rho_{1,\mathrm{st}}$ is directly obtained from the expression of $w_1(X)$ and (4.13):

$$\rho_{1,\mathrm{st}}(X) = C\phi_1(X)^2 e^{-vX} \tag{4.14}$$

where C is a normalization constant such that $\int_0^\infty \rho_{1,\mathrm{st}}(X)dX = 1$. The density $\rho_{1,\mathrm{st}}$ depends only on the slowest eigenvector ϕ_1 and its shape near the critical velocity will be studied in Sect. 5.1.

The extinction probability of A_0 particles in the modified process is given by construction by $\tilde{R}_0(x,t,t';\infty) = Q_e(x,t)/Q_e(x,t+t')$ and thus becomes $Q_e(x,t)/Q_e^*(x)$ as $t' \to \infty$ in the quasi-stationary regime. Therefore a particle A_0 gets extinct with probability 1 in the long time limit. Thus particles A_0 are produced by the particle A_1 and later on are absorbed by the wall. Their stationary density $\rho_{0,\mathrm{qs}}$ satisfies:

$$\partial_X^2 \rho_{0,\mathrm{st}}(X) - \partial_X \left(w_0(X)\rho_{0,\mathrm{st}}(X)\right) + \sum_k \beta_k Q_e^*(X)^{k-1}(k-1)\rho_{0,\mathrm{st}}(X)$$
$$= -g''\left(Q_e^*(X)\right) Q_e^*(X)\rho_{1,\mathrm{st}}(X). \tag{4.15}$$

The change of variable $\rho_{0,\mathrm{st}}(X) = e^{-vX} Q_e^*(X)\psi(X)$ leads for ψ to an equation of the type:

$$\mathcal{L}[\psi] = K(X) \tag{4.16}$$
$$K(X) = -g''\left(Q_e^*(X)\right)\rho_{1,\mathrm{st}}(X)e^{vX} \tag{4.17}$$

where $\mathcal{L}$ is the linear operator defined in (3.8). This equation is an inhomogeneous second order linear differential equation that can be solved easily (because one solution of the homogeneous equation, $\partial_x Q_e^*(x)$, is known). The general solution of (4.16) is

$$\rho_{0,\mathrm{st}}(X) = e^{-vx} Q_e^*(X)\partial_x Q_e^*(X) \int_a^X \frac{dy}{[\partial_x Q_e^*(y)]^2 e^{-vy}} \int_b^y \partial_x Q_e^*(z) e^{-vz} K(z)dz.$$

The stationary profile $\rho_{0,\mathrm{st}}(X)$ is then obtained by choosing $a=0$ and $b \to \infty$ for $\rho_{0,\mathrm{st}}(X)$ to vanish at $X=0$ and $X \to \infty$:

$$\rho_{0,\mathrm{st}}(X) = e^{-vX} Q_e^*(X)\partial_x Q_e^*(X)$$
$$\times \int_0^X \frac{dy}{[\partial_x Q_e^*(y)]^2 e^{-vy}} \int_y^\infty \partial_x Q_e^*(z) g''\left(Q_e^*(z)\right)\rho_{1,\mathrm{st}}(z)dz. \tag{4.18}$$

Equations (4.11, 4.14, 4.18) give the exact average quasi-stationary profile in terms of only Q_e^* and ϕ_1 which have been used for the construction of the quasi-stationary state. Section 5 will describe the limit $v \to v_c$ of $Q_e^*(x)$, $\phi_1(x)$ and $\rho_{\mathrm{qs}}(X)$.

The results (4.11, 4.14, 4.18) could also be derived from the dynamical system approach of Sect. 3.3. The average quasi-stationary profile $\rho_{\mathrm{qs}}(X)$ at distance X from the wall is obtained by considering the test function $f(x) = \mu\delta(x-X)$ and taking the μ-derivative of (3.27) at $\mu = 0$, as for the average size in (3.32). Relations (3.31a, b) between $\mathcal{A}^{(1)}[\phi_i]$, $\mathcal{A}^{(2)}[\phi_i,\phi_j]$, $\mathcal{L}$ and $\mathcal{F}^{(2)}[\phi_i,\phi_j]$ are then exactly equivalent to the differential equations (4.12, 4.13) in their stationary regime.

5 Universality Near in the Critical Velocity

In this section, we determine the scaling properties of the quasi-stationary profile when $v_c - v \to 0^+$. We are going to see that, in this limit, everything can be expressed in terms of the shape $Q_{v_c}(x)$ of a front on the infinite line at the critical velocity.

5.1 Extinction Probability and Relaxation Times

The stable fixed point $Q_e^*(x)$ of the functional $\mathcal{F}$ defined in (3.3) satisfies the differential equation (3.5) with the boundary condition $Q_e^*(0)=1$ and $Q_e^*(x)$ goes to 0 or 1 as $x\to\infty$.

In Appendix B, we use a perturbation theory to analyze (3.5) on the infinite line by $v_c - v \to 0^+$ and to obtain the extinction probability $Q_e^*(x)$ in presence of the wall in the same limit. The solution Q_{v_c} of (3.5) on the infinite line such that $Q_{v_c}(x_0)=1/2$ (where x_0 is some arbitrary position), $Q_{v_c}(x)\to 1$ as $x\to -\infty$ and $Q_{v_c}(x)\to 0$ as $x\to +\infty$ is given for large negative x:

$$Q_{v_c}(x) = 1 + (A_c x + B_c)e^{v_c x/2} + O\left(e^{v_c x}\right) \tag{5.1}$$

where A_c and B_c are constant depending on x_0. For v close v_c, one can show (see Appendix B) that, on a domain of length L that we will call region I, $Q_e^*(x)$ is given by:

$$Q_e^*(x) \underset{\text{region I}}{\simeq} 1 - \frac{A_c L}{\pi}\sin\left(\frac{\pi x}{L}\right)\exp\left[\frac{v_c}{2}(x - L - B_c/A_c)\right] \tag{5.2}$$

where A_c and B_c are the constants defined as in (5.1). The length L is given by:

$$L = 2\pi/\sqrt{v_c^2 - v^2} \underset{v\to v_c}{\sim} (v_c - v)^{-1/2} \tag{5.3}$$

and diverges at the critical velocity. In the domain $x \gtrsim L$ (region II), the non-linearities of (3.5) must be taken into account and Q_e^* is given at leading order by:

$$Q_e^*(x) \underset{\text{region II}}{=} Q_{v_c}(x - L - B_c/A_c) + O(1/L^2). \tag{5.4}$$

The relaxation times τ_n of the dynamics (3.2) near the stable fixed point Q_e^* are related to the eigenvalues $\lambda_n = -1/\tau_n$ of the linear operator (3.8). An eigenvector ϕ_n with eigenvalue λ_n satisfies the second order differential equation:

$$\partial_x^2\phi_n - v\partial_x\phi_n + g'\left(Q_v(x)\right)\phi_n = \lambda_n\phi_n \tag{5.5}$$

with the boundary conditions:

$$\phi_n(0) = 0, \quad \phi_n(x) \xrightarrow{x\to+\infty} 0. \tag{5.6}$$

As for Q_e^*, the shape of the eigenvectors ϕ_n can be obtained from the ones on the infinite line as shown in appendix (B). The effect of the boundary condition (5.6) on the wall is to select the ones which vanish at $x=0$. A perturbative expansion in $v_c - v$ and λ shows that the first eigenvalues λ_n in presence of the absorbing wall are given by:

$$\lambda_n \simeq -\frac{(n^2-1)\pi^2}{L^2} - \frac{4n^2\pi^2}{v_c L^3} - \frac{12n^2\pi^2}{v_c^2 L^4} + O\left(\frac{1}{L^5}\right). \tag{5.7}$$

One can notice that the first three terms in this expansion are independent of the precise form $g(Q)$ of the non-linearities. The eigenvector ϕ_n associated to λ_n is given (up to a multiplicative constant) at leading order by:

$$\phi_n(x) \underset{\text{region I}}{\simeq} \frac{A_c v_c L(-1)^{n-1}}{2n\pi}\sin\left(\frac{n\pi x}{L}\right)\exp\left[\frac{v_c}{2}\left(x - L - \frac{B_c}{A_c}\right)\right] \tag{5.8a}$$

in the region I of size L near the wall and by:

$$\phi_n(x) \underset{\text{region II}}{\simeq} \partial_x Q_{v_c}(x - L - B_c/A_c) \tag{5.8b}$$

in the region II such that $x \gtrsim L$. Appendix B also gives the order of magnitude of the first correction to ϕ_n in both regimes $x \ll L$ and $x > L$. As the relaxation times are given by $\tau_n = -1/\lambda_n$, the longest relaxation time τ_1 is given (5.7) by

$$\tau_1 \simeq \frac{L^3 v_c}{4\pi^2} \sim \frac{\pi}{\sqrt{2v_c}}(v_c - v)^{-3/2} \tag{5.9}$$

and is much larger than all the other τ_n's ($n \geq 2$). One can notice that the same expression of τ_1 was obtained in [12] by a very different approach.

5.2 Quasi-Stationary Profile

The quasi-stationary profile (4.11) is the sum of two contributions (4.14, 4.18) of the modified process. For v close to v_c, the normalization constant C in (4.14) is dominated by the contribution of region I and is given at lowest order by $C \simeq 8\pi^2 e^{v_c(L+B_c/A_c)}/(A_c^2 v_c^2 L^3)$. Thus the stationary profile $\rho_{1,\mathrm{st}}(X)$ at distance X from the wall in the modified process is given by:

$$\rho_{1,\mathrm{st}}(X) \underset{\text{region I}}{\simeq} \frac{2}{L} \sin\left(\frac{\pi X}{L}\right)^2 \tag{5.10}$$

for $X < L$. For $X \gtrsim L$ (region II), the density is given by:

$$\rho_{1,\mathrm{st}}(X) \underset{\text{region II}}{\simeq} \frac{8\pi^2 e^{v_c(L+B_c/A_c)}}{A_c^2 v_c^2 L^3} e^{-vX} \left[\partial_x Q_{v_c}(X - L - B_c/A_c)\right]^2. \tag{5.11}$$

The corrections to this profile are of order $1/L$ in both regions.

The density profile $\rho_{0,\mathrm{st}}(X)$ given by (4.18) can be rewritten in the following way:

$$\rho_{0,\mathrm{st}}(X) = e^{-vX} Q_e^*(X) \partial_x Q_e^*(X) \int_0^X \frac{dy}{[\partial_x Q_e^*(y)]^2 e^{-vy}} (I_\infty - I(y)) \tag{5.12}$$

where

$$I(y) = \int_0^y \partial_x Q_e^*(z) g''(Q_e^*(z)) \rho_{1,\mathrm{st}}(z) dz \tag{5.13}$$

and $I_\infty = \lim_{y\to+\infty} I(y)$. For $x < L$, $\phi_1(x)$ and $\partial_x Q_e^*$ are of order $e^{v_c(x-L)/2}$ and Q is of order 1; for $x - L \gg 1$ (region II), all the terms in the integral (5.13) decrease exponentially. Thus, the integral is dominated by the region of size of order 1 near $x \simeq L$ and I_∞ scales for $v_c - v \to 0^+$ as:

$$I_\infty \simeq \frac{1}{L^3}\left[\frac{8\pi^2}{A_c^2 v_c^2} \int_{-\infty}^{+\infty} [\partial_x Q_{v_c}(z)]^3 g''(Q_{v_c}(z)) e^{-v_c z} dz\right]. \tag{5.14}$$

One notices that I_∞ is negative since $Q_e^*(x)$ is a decreasing function and that I_∞ is independent of the choice of the reference x_0 such that $Q_{v_c}(x_0) = 1/2$. On the other hand, for the same reasons, $I(y)$ is exponentially small and can be neglected compared to I_∞ in the linear region $0 < y < L$.

At leading order in the region I, one gets from (5.12):

$$\rho_{0,\mathrm{st}}(X) \simeq I_\infty e^{-v_c X} \partial_x Q_e^*(X) \int_0^X \frac{1}{[\partial_x Q_e^*(y)]^2 e^{-vy}} dy. \tag{5.15}$$

From (5.2), $\partial_x Q_e^*(x)$ is given for $x < L$ by:

$$\partial_x Q_e^*(x) \underset{\text{region I}}{\simeq} -\frac{A_c L}{\pi} \left[\frac{v_c}{2} \sin\left(\frac{\pi x}{L}\right) + \frac{\pi}{L} \cos\left(\frac{\pi x}{L}\right)\right] e^{\frac{v_c}{2}(x-L-B_c/A_c)}$$

up to exponentially small corrections. The integral in (5.15) is equal to:

$$\begin{aligned}\int_0^X \frac{1}{[\partial_x Q_e^*(y)]^2 e^{-vy}} dy &\simeq \frac{\pi^2 e^{v_c(L+B_c/A_c)}}{A_c^2 L^2} \int_0^X \frac{dy}{[\frac{v_c}{2}\sin(\frac{\pi y}{L}) + \frac{\pi}{L}\cos(\frac{\pi y}{L})]^2} \\ &\simeq \frac{e^{v_c(L+B_c/A_c)}}{A_c^2} \frac{\sin(\frac{\pi X}{L})}{\frac{v_c}{2}\sin(\frac{\pi X}{L}) + \frac{\pi}{L}\cos(\frac{\pi X}{L})}.\end{aligned}$$

Finally the density of A_0 particles is given in the region I by:

$$\rho_{0,\mathrm{st}}(X) \simeq -\frac{I_\infty L e^{\frac{v_c B_c}{2A_c}}}{A_c \pi} \sin\left(\frac{\pi X}{L}\right) \exp\left[-\frac{v_c}{2}(X-L)\right]. \tag{5.16}$$

In the region $x < L$, the number of particles A_0 is exponentially large and $\rho_{1,\mathrm{st}}(X)$ is bounded and much smaller than $\rho_{0,\mathrm{st}}(X)$ as shown in (5.10, 5.16). Thus the quasi-stationary average profile $\rho_{\mathrm{qs}}(X)$ scales in the linear domain as:

$$\rho_{\mathrm{qs}}(X) \underset{\text{region I}}{\simeq} \frac{K v_c^2}{4\pi L^2} e^{-v_c(X-L)/2} \sin\left(\frac{\pi X}{L}\right) \tag{5.17}$$

with a constant K given by:

$$K = -\frac{32\pi^2 e^{\frac{1}{2}v_c B_c/A_c}}{A_c^3 v_c^4} \int_{-\infty}^{+\infty} [\partial_x Q_{v_c}(z)]^3 g''(Q_{v_c}(z)) e^{-v_c z} dz. \tag{5.18}$$

One can check that this expression of K is independent of the reference point x_0 chosen for Q_{v_c}. For $X - L$ of order 1, $\rho_{\mathrm{qs}}(X)$ is of order $1/L^3$ and in the domain $X > L$ (region II), the density decreases exponentially, so that the total number of individuals is dominated by the region $0 < X < L$. The average size in the quasi-stationary regime is thus given by:

$$\begin{aligned}\langle N\rangle_{\mathrm{qs}} &= \int_0^\infty \rho_{\mathrm{qs}}(X) dX \simeq \frac{K}{L^3} e^{v_c L/2} \\ &\sim (v_c - v)^{3/2} \exp\left[\pi\sqrt{\frac{v_c}{2}}(v_c - v)^{-1/2}\right].\end{aligned} \tag{5.19}$$

The divergence (5.19) of $\langle N\rangle_{\mathrm{qs}}$ can also be obtained from (3.32) by expanding Q_e^* in the basis of the eigenvectors and using (3.31a, b): the leading term (5.19) could then be obtained by truncating the expansion of Q_e^* after the first eigenvectors ϕ_1, as the first eigenvalue $\lambda_1 \simeq (v_c - v)^{-3/2}$ is much smaller than the next ones that scale as $(v_c - v)^{-1}$.

In [7, 9], Brunet et al. considered a population whose size N is kept constant by always selecting the N individuals with the largest x_i. From [8], they predicted that the effect of the finite size N is to shift the average velocity $\langle v\rangle$ of the population by an amount:

$$\langle v\rangle = v_c - \frac{v_c\pi^2}{2\log^2 N} + (v_c\pi^2)\frac{3\log\log N}{\log^3 N} + \dots. \tag{5.20}$$

This N dependence of the velocity was recently derived rigorously in [33]. It is interesting to notice that this relation between $\langle v\rangle - v_c$ and N is the same as (5.19) between $\langle N\rangle$ and $v - v_c$ in the quasi-stationary state of the present model with an absorbing wall.

6 Exponential Model

There is a simplified version of the problem of a branching random walk with an absorbing wall, the exponential model, which can be solved exactly [7, 9]. In this exponential model, time is discrete and at each generation, each individual is replaced by its offspring distributed according to a Poisson point process: if the parent is at position x, then for every interval $[y, y+dy]$ there is an offspring in this interval with probability $\psi(y-x)dy$.

By analyzing the first time step one can show that the extinction probability $Q_e(x,t)$ at time t of the descendance of an individual located at distance x from the wall at time $t=0$ evolves according to:

$$Q_e(x,t+1) = \mathcal{F}(Q_e(x,t)) = \exp\left(-\int_0^\infty \psi(y+v-x)\,(1-Q_e(y,t))\,dy\right). \tag{6.1}$$

For a general ψ, $\mathcal{F}$ is a functional of Q_e for which one could try to generalize the approaches of Sects. 3 and 4. For the special case $\psi(x) = e^{-x}$, however, the problem reduces to the study of a one-dimensional mapping. For any $Q_e(x,0)$, it is easy to see from (6.1) that at $t \geq 1$ $Q_e(x,t)$ takes the form

$$Q_e(x,t) = e^{-c_t e^x} \tag{6.2}$$

and depends on a single parameter c_t which satisfies the following recursion:

$$c_{t+1} = F(c_t) \tag{6.3}$$

$$F(c) = e^{-v}\left(1 - \int_0^\infty e^{-y} e^{-ce^y}\,dy\right). \tag{6.4}$$

For all v, there is an attractive fixed point $c_* > 0$ and the population always has a non-zero survival probability $1 - Q_e^*(x) = 1 - e^{-c_* e^x}$ in the long time limit. The relaxation of $Q_e(x,t)$ to $Q_e^*(x)$ is entirely controlled by the relaxation of c_t to c_*: there is a unique eigenvalue $\Lambda = F'(c_*) < 1$ with the eigenvector

$$\phi_1(x) = -\exp(x - c_* e^x) \tag{6.5}$$

and one has $Q_e(x,t) \simeq Q_e^*(x) - K\Lambda^t e^{x-c_* e^x}$. Generating functions G_1 and G_2 defined as in (3.11, 3.14) also satisfy the same dynamics (6.1). A general initial condition $Q(x,0) = Q_0(x)$ is mapped at the first time step to a function $Q(x,1) = e^{-c(Q_0)e^x}$ where the coefficient $c(Q_0)$ is defined by

$$c(Q_0) = e^{-v}\left(1 - \int_0^\infty e^{-y} Q_0(y)dy\right) \tag{6.6}$$

and has the following long time behaviour:

$$Q(x,t) = \exp(-c_t e^x) \simeq Q_e^*(x) - \mathcal{B}(c(Q_0))\Lambda^{t-1} e^{x-c_* e^x} \tag{6.7}$$

with an amplitude $\mathcal{B}(c)$ similar to the function $\mathcal{A}$ of Sect. 2. As in (2.17), the function $\mathcal{B}(c)$ satisfies:

$$\Lambda\mathcal{B}(c) = \mathcal{B}(F(c)), \qquad \mathcal{B}'(c_*) = 1 \tag{6.8}$$

with F given in (6.4).

The reduction to a one-parameter family $\exp(-ce^x)$ makes the analysis of this spatial model similar to what we did in Sect. 2.

For example, the generating function of the number $N(X)$ of individuals at a distance greater than a given X from the wall is given in the quasi-stationary regime by (3.28) with $\mathcal{A}(Q) = \mathcal{B}(c(Q))/\Lambda$ and $f(x) = \mu\theta_X(x) \equiv \mu\theta(x - X)$. For the exponential model, it can be reduced to a simple function of μ:

$$\langle e^{-\mu N(X)}\rangle_{\text{qs}} = \frac{1}{\Lambda}\frac{d}{ds}\mathcal{B}\left(c\left((Q_e^* + s\phi)e^{-\mu\theta_X}\right)\right)\Big|_{s=0}.$$

By decomposing $c((Q_e^* + s\phi_1)e^{-\mu\theta_X})$ in the following way:

$$\begin{aligned}
c\left((Q_e^* + s\phi_1)e^{-\mu\theta_X}\right) &= e^{-v}\left(1 - \int_0^\infty e^{-y}\left(Q_e^*(y) + s\phi_1(y)\right)dy\right)\\
&\quad + e^{-v}(1-e^{-\mu})\int_X^\infty e^{-y}\left(Q_e^*(y) + s\phi_1(y)\right)dy\\
&= c_* + s\Lambda + (1-e^{-\mu})\left(J_0(X) + s\Lambda J_1(X)\right)
\end{aligned}$$

$$J_0(X) = e^{-v}\int_X^\infty e^{-y-c_* e^y}\,dy \tag{6.9}$$

$$J_1(X) = \frac{1}{\Lambda}\int_X^\infty e^{-c_* e^y}\,dy \tag{6.10}$$

where one can verify that $J_1(0) = 1$, one obtains the generating function:

$$\langle e^{-\mu N(X)}\rangle_{\text{qs}} = \left(1 - (1-e^{-\mu})J_1(X)\right)\mathcal{B}'(c_* + (1-e^{-\mu})J_0(X)). \tag{6.11}$$

It follows that the average density profile $\langle\rho(X)\rangle_{\text{qs}}$ obtained by taking the derivative of $\langle N(X)\rangle_{\text{qs}}$ and the average size of the population are given by:

$$\langle\rho(X)\rangle_{\text{qs}} = -J_1'(X) + \mathcal{B}^{(2)}(c_*)J_0'(X) \tag{6.12}$$

$$\langle N\rangle_{\text{qs}} = 1 + \left(-\mathcal{B}^{(2)}(c_*)\right)J_0(0). \tag{6.13}$$

Let us now analyze the large v limit (which for the exponential model plays the role [9] of the $v \to v_c$ limit of Sect. 5). The mapping $F(c)$ has the following expansion near $c = 0$ (corresponding to $Q_e^*(x) = 1$):

$$F(c) = e^{-v}\left(-c\ln c + (1-\gamma)c + \frac{c^2}{2} - \frac{c^3}{12} + \dots\right) \tag{6.14}$$

where $\gamma = -\Gamma'(1)$ is Euler's constant. For large v, the stable fixed point c_* of F is given by

$$c_* \simeq e^{1-\gamma} \exp(-e^v)$$

and $\Lambda \simeq 1 - e^{-v}$. As in Sect. 2, one can compute $\mathcal{B}^{(2)} \simeq -1/c_*$ from (6.8). The functions J_0 and J_1 in (6.9, 6.10) have different shapes for $X < L$ (region I) and $X > L$ (region II) where the length L is given by:

$$L = -\ln c_* \simeq e^v. \tag{6.15}$$

Thus the leading terms in (6.12, 6.13) when $X < L$ (region I) correspond to J_0 contributions and, for $v \to \infty$, the average quasi-stationary profile and sizes are given by:

$$\langle \rho(X) \rangle_{\text{qs}} \simeq \frac{1}{L} e^{-(X-L)} \quad \text{(region I, } X < L) \tag{6.16}$$

$$\langle N \rangle_{\text{qs}} \simeq \frac{e^{-v}}{c_*} \sim \frac{1}{L} e^L. \tag{6.17}$$

In region II (for $X > L$), $\langle \rho(X) \rangle_{\text{qs}}$ is decreasing as $e^{-c_* e^X}$. As in Sect. 5, the region of length L near the wall has the dominant contributions to the size of the population. Also as in Sect. 4, one could interpret the J_1 terms as the contribution of the A_1 particle and the J_0 terms as contributions of the A_0 particles in the context of the modified process.

One can notice that, in the exponential model too, (6.17) yields the same relation between the size of the population and the velocity as (39) in [9] (where the size of the population is kept constant but the velocity fluctuates).

In the limit $v \to \infty$, it is possible to obtain the whole generating function $\langle e^{-\mu N} \rangle_{\text{qs}}$. In this limit, N scales as $1/c_*$ and one should take μ of order c_* and thus $\mathcal{B}(c)$ needs to be known for $c - c_* = O(c_*)$. If $F(c)$ in (6.14) is truncated after the first two leading terms near c_* as in Sect. 2.3 and Appendix A, one obtains $\mathcal{B}(c_* + c_* u) \simeq \ln(1+u)$ in the limit $v \to \infty$ and the following generating function of the size:

$$\left\langle e^{-\mu N / \langle N \rangle_{\text{qs}}} \right\rangle \underset{v \to \infty}{\simeq} \frac{1}{1+\mu}. \tag{6.18}$$

Thus in the quasi-stationary regime the size of the population has an exponential distribution in contrast to what happens (2.33) in the Galton-Watson process. In particular, the ratio $\langle N^2 \rangle_{\text{qs}} / \langle N \rangle_{\text{qs}}^2$ goes to 2: this result is similar to numerical results for lattice branching random walks obtained in [12].

7 Conclusion

In the present work, we have studied the quasistationary regime of a branching random walk in presence of an absorbing wall below the critical velocity [12]. To do so, we have developed two methods. The first one, discussed in Sect. 3, is a dynamical system approach allowing to determine the generating functions of the size of the population.

In the second approach, one constructs from the original process a modified stochastic process equivalent to the quasi-stationary regime (Sect. 4). This construction requires the knowledge of $P_0(x,t) = Q_e(x,t)$ and $P_1(x,t)$, the probabilities of observing 0 and 1 survivor at time t for an initial individual at x. These methods allowed us to determine (5.17–5.19) the average population size and the density profile in the quasi-stationary regime near v_c. The average profile has a universal shape (5.17) in a domain of size $L \sim (v_c - v)^{-1/2}$ and the average size diverges as in (5.19) in a universal way which does not depend on the precise form of the branching rates β_k. These quantities have also been obtained for the exponential model in Sect. 6. We noticed that both for the branching random walk and the exponential model, the relation between the velocity and the size of the population (5.19, 6.17) is the same as in a model recently studied in [9] for a population of constant size.

Beyond the average quasistationary profile which has a universal shape near v_c (i.e. is independent of the precise form of the non-linearities $g(q)$ in (3.4)), it would be interesting to see whether the whole quasi-stationary measure is universal when $v \to v_c$. In particular, it would be interesting to know whether the size of the population has an exponential distribution in the quasi-stationary regime as in the exponential model (6.18).

Our approach was limited to the case $v < v_c$. For $v > v_c$ one expects [12] that there is no quasi-stationary regime. The construction of the modified process of Sect. 4 remains however valid and this could be a starting point to understand the absence of a quasi-stationary regime.

More generally, the construction of the modified process is valid as long as the individuals are independent. In particular, a modified process could be constructed to describe the evolution of a population conditioned to a finite size $N_T = 1$ in non-uniform media, as in [3, 4], where the branching rates, the drift and the diffusion coefficients depend on the position x and where the domains have more complicated geometries (d-dimensional with absorption on the boundaries). This could be a way of understanding the effect of the geometry of the domain on the existence and on the properties of a quasi-stationary regime.

Using the modified process, one could also study the dynamical properties and the correlations in time of the quasi-stationary regime. In particular, one could try to determine the statistical properties of the genealogies in this quasi-stationary regime and compare them with the results obtained recently for a population of fixed size [7, 9].

Appendix A: Universal Distributions of Population Sizes in the Quasistationary Regime Near the Transition in Birth-Death Processes: The Case of Slowly Decreasing Branching Rates

In this appendix, we discuss briefly the quasi-stationary regime of the Galton-Watson process when the branching rates β_k decay too slowly for $\overline{k^2}$ to be finite. We consider the case where for large k

$$\beta_k \simeq B k^{-1-\eta} \tag{A.1}$$

with $1 < \eta < 2$. Then, for Q close to 1, $F(Q)$ takes the form for Q close to 1:

$$F(Q) \simeq \alpha(Q-1) + C(1-Q)^{\eta} + \dots \tag{A.2}$$

with $C = B\Gamma(-\eta)$ and $\alpha = \sum_k (k-1)\beta_k$.

For $\alpha < 0$, the fixed point $Q_e^* = 1$ is attractive and one gets from (2.30) for Q close to 1:

$$\mathcal{A}(Q) \simeq \frac{(Q-1)}{(1 - \frac{C}{\alpha}(1-Q)^{\eta-1})^{1/(\eta-1)}}. \tag{A.3}$$

For small $\alpha > 0$, the fixed point becomes $1 - Q_e^* \simeq (\alpha/C)^{1/(\eta-1)}$, the eigenvalue $F'(Q_e^*) = \lambda$ scales as $\lambda \simeq -\alpha(\eta - 1)$ and by integrating (2.30), one gets:

$$\mathcal{A}(Q) \simeq (1-Q)^{1-\eta}\left(1 - \frac{C}{\alpha}(1-Q)^{\eta-1}\right)\frac{1}{\eta-1}\left(\frac{\alpha}{C}\right)^{\frac{\eta}{\eta-1}}. \tag{A.4}$$

Then by taking μ small in (2.16), one gets for $\alpha < 0$:

$$\langle e^{-\mu N}\rangle_{\text{qs}} \simeq \left(\frac{1}{1 - \frac{C}{\alpha}\mu^{\eta-1}}\right)^{\frac{\eta}{\eta-1}} \tag{A.5}$$

whereas for $\alpha > 0$, one gets:

$$\langle e^{-\mu N}\rangle_{\text{qs}} \simeq \left(1 + \mu\left(\frac{C}{\alpha}\right)^{1/(\eta-1)}\right)^{-\eta}. \tag{A.6}$$

Expressions (A.5, A.6), valid for $|\alpha|$ small, show that both below and above the transition, the distributions of N in the quasi-stationary regime are universal (as up to a rescaling they depend only on η) and that they differ from what was found (2.33) in the case $\eta > 2$).

Appendix B: Perturbative Calculation of the Shape of the Front, of the Eigenvalues and Eigenvectors for v Close to v_c

In this appendix we determine perturbatively, for $v - v_c$ small, the stable solution Q_e^* of (3.5), the eigenvalues λ_n and their associated eigenvectors ϕ_n of the linear operator $\mathcal{L}$ defined in (3.8).

For the shape of Q_e^* as for the eigenvectors our main idea is to derive a perturbation theory in powers of $v_c - v$ (and also in powers of λ for the eigenvectors) in the region where the non-linear terms cannot be neglected (i.e. in the region where $1 - Q_e^*$ is not small) and to match this expansion with the solution which can be obtained in the linear region (where $1 - Q_e^* \ll 1$).

B.1 The Shape of Q_e^*

It is convenient to consider the solution $Q_v(x)$ of (3.5)

$$Q_v'' - vQ_v' + g(Q_v) = 0 \tag{B.1}$$

on the infinite line such that $Q_v(x) \to 1$ as $x \to -\infty$ and $Q_v(x) \to 0$ as $x \to +\infty$ (instead of the solution $Q_e^*(x)$ on the semi infinite line). Here $g(Q) = \sum_k \beta_k(Q^k - Q)$. The boundary conditions at $\pm\infty$ determine Q_v up to a translation. One particular solution may be selected by imposing that

$$Q_v(x_0) = \frac{1}{2} \tag{B.2}$$

where x_0 is a fixed position (which can be chosen arbitrarily and of course will play no role in our final results). For $v < v_c = 2\sqrt{g'(1)}$, the solution of (B.1) has damped oscillations, as $x \to -\infty$, of the form

$$Q_v(x) = 1 + U_v \sin\left(\frac{\pi(x + \Phi_v)}{L}\right)\exp\left(\frac{vx}{2}\right) + O\left(e^{vx}\right) \tag{B.3}$$

where the length L is defined by:

$$L = 2\pi(v_c^2 - v^2)^{-\frac{1}{2}} \tag{B.4}$$

so that for large L, one gets

$$v \simeq v_c - \frac{2\pi^2}{v_c L^2}. \tag{B.5}$$

The constants U_v and Φ_v in (B.3) are a priori complicated functions of v (or L) and also depend on x_0.

If y_L is the position of the right-most zero of $1 - Q_v(x)$, then the solution Q_e^* of (3.5) we are looking for is given by:

$$Q_e^*(x) = Q_v(x + y_L). \tag{B.6}$$

As v approaches v_c, the length L diverges, and (B.3) allows one to estimate y_L up to corrections exponentially small in L

$$y_L \simeq -L - \Phi_v. \tag{B.7}$$

We now assume that the solution Q_{v_c} of (B.1) is known for $v = v_c$. One can then expand $Q_v(x)$ in powers of $\frac{1}{L}$ or in powers of $v_c - v$

$$Q_v = Q_{v_c} + \frac{1}{L^2} R + \ldots = Q_{v_c} + (v_c - v)\frac{v_c}{2\pi^2} R + \ldots. \tag{B.8}$$

Putting (B.8) into (B.1) one gets that R should satisfy

$$R'' - v_c R' + g'(Q_{v_c})R = -\frac{2\pi^2}{v_c} Q'_{v_c}. \tag{B.9}$$

One can solve this equation with the required boundary conditions (the equations leading to higher order terms in the expansion could be solved as well) and one gets:

$$R(x) = \frac{2\pi^2}{v_c} Q'_{v_c}(x) \int_{x_c}^{x} \frac{dy}{[Q'_{v_c}(y)]^2 e^{-vy}} \int_{y}^{+\infty} [Q'_{v_c}(z)]^2 e^{-vz} dz. \tag{B.10}$$

For $x \to -\infty$, one can show from (B.1) that

$$Q_{v_c}(x) = 1 + (A_c x + B_c) e^{\frac{v_c x}{2}} + H_2(x) e^{v_c x} + H_3(x) e^{\frac{3 v_c x}{2}} + \ldots \tag{B.11}$$

where the coefficients A_c and B_c depend on x_0 and the polynomes H_n could be obtained explicitly in terms of these coefficients A_c and B_c by analyzing the non-linear terms of (B.1).

One can also show from (B.10) or (even better) directly from (B.9) that for $x \to -\infty$:

$$R(x) = \pi^2 \left(-\frac{A_c x^3}{6} - \frac{B_c x^2}{2} - \frac{A_c x^2}{v_c} + Cx + D \right) e^{\frac{v_c x}{2}} + O\left(e^{v_c x}\right) \tag{B.12}$$

where the constants A_c and B_c are the same as in (B.11) and the constants C and D could be determined from the knowledge of $Q_{v_c}(x)$ and (B.10).

This leads (B.8, B.11, B.12) for large negative x to

$$Q_v(x) = 1 + \left[A_c x + B_c + \frac{\pi^2}{L^2}\left(-\frac{A_c x^3}{6} - \frac{B_c x^2}{2} - \frac{A_c}{v_c}x^2 + Cx + D\right) + \dots\right] e^{\frac{v_c x}{2}} + O(e^{v_c x}) \tag{B.13}$$

On the other hand for large negative x, in the range $1 \ll |x| \ll L$, expression (B.3) becomes

$$Q_v(x) = 1 + U_v\left[\frac{\pi(x+\Phi_v)}{L} - \frac{\pi^3(x+\Phi_v)^3}{6L^3} - \frac{\pi^3 x(x+\Phi_v)}{v_c L^3} + O\left(L^{-5}\right)\right] e^{\frac{v_c x}{2}} + \dots. \tag{B.14}$$

The comparison of (B.13) and (B.14) leads to the following expansions for U_v and Φ_v:

$$U_v = \frac{L}{\pi}A_c + \frac{\pi}{L}\left(C + \frac{B_c}{v_c} + \frac{B_c^2}{2A_c}\right) + O\left(\frac{1}{L^3}\right) \tag{B.15a}$$

$$\Phi_v = \frac{B_c}{A_c} + \frac{\pi^2}{L^2}\left(\frac{D}{A_c} - \frac{B_c^3}{3A_c^3} - \frac{B_c^2}{A_c^2 v_c} - \frac{B_c C}{A_c^2} + O\left(\frac{1}{L^4}\right)\right). \tag{B.15b}$$

Then using (B.6, B.7) one gets the following expressions for $Q_e^*(x)$ in the region where $L - x \gg 1$

$$Q_e^*(x) = 1 - \frac{A_c L}{\pi}\exp\left[\frac{v_c(x - L - \frac{B_c}{A_c})}{2}\right]\left[\sin\left(\frac{\pi x}{L}\right) + O\left(\frac{1}{L^2}\right)\right] + O\left(e^{v_c(x-L)}\right) \tag{B.16}$$

and in the region $x > L$ or $x - L = O(1)$:

$$Q_e^*(x) = Q_{v_c}\left(x - L - \frac{B_c}{A_c}\right) + O\left(\frac{1}{L^2}\right). \tag{B.17}$$

We emphasize that the knowledge of $Q_{v_c}(x)$ is sufficient to determine all the higher order corrections in $\frac{1}{L}$ expansions.

B.2 The Eigenvalues and the Eigenvectors ϕ_λ

We consider now the shape of the eigenvectors $\phi_{\lambda,v}$ of the linear operator $\mathcal{L}$ on the infinite line. On the infinite line, to each λ, one can associate an eigenvector $\phi_{\lambda,v}$ which satisfies:

$$\phi''_{\lambda,v} - v\phi'_{\lambda,v} + g'(Q_v)\phi_{\lambda,v} = \lambda\phi_{\lambda,v} \tag{B.18}$$

with the boundary conditions (5.6) $\phi_{\lambda,v}(x) \to 0$ as $x \to \pm\infty$. For $v < v_c$, the solution when $x \to -\infty$ has the form

$$\phi_{\lambda,v}(x) = V_{\lambda,v}\sin\left(\frac{\pi(x + \Psi_{\lambda,v})}{L_{\lambda,v}}\right)\exp\left(\frac{vx}{2}\right) + O(e^{vx}) \tag{B.19}$$

where the length $L_{\lambda,v}$ is defined by

$$L_{\lambda,v} = 2\pi(v_c^2 - 4\lambda - v^2)^{-\frac{1}{2}} = L\left(1 - \frac{L^2\lambda}{\pi^2}\right)^{-\frac{1}{2}}. \tag{B.20}$$

On the other hand, it is easy to see that $Q'_{v_c}(x)$ is an eigenvector for $\lambda = 0$ and $v = v_c$, (in fact for any v, one has $\phi_{0,v}(x) = Q'_v(x)$ up to a multiplicative constant) and one can try to expand $\phi_{\lambda,v}$ in powers of λ and $v_c - v$

$$\phi_{\lambda,v}(x) = Q'_{v_c}(x) + \sum_{n+m\geq 1} S_{nm}(x)\frac{\lambda^n}{L^{2m}}. \tag{B.21}$$

The S_{nm}'s could be determined recursively solving inhomogeneous equations similar to (B.9). For example

$$S''_{10} - v_c S'_{10} + g'(Q_{v_c})S_{10} = Q'_{v_c}$$

$$S''_{01} - v_c S'_{01} + g'(Q_{v_c})S_{01} = -\frac{2\pi^2}{v_c}Q''_{v_c} - g''(Q_{v_c})RQ'_{v_c}.$$

As in (B.13) one gets that for large negative x

$$\phi_{\lambda,v}(x) = \left[\frac{A_c v_c}{2}x + \frac{B_c v_c}{2} + A_c + O(\lambda) + O\left(\frac{1}{L^2}\right)\right]e^{\frac{v_c x}{2}} + O\left(e^{v_c x}\right). \tag{B.22}$$

Comparing (B.22) with (B.19) in the range of large negative x with $x \ll L_{\lambda,v}$ (as in (B.13, B.14)) one gets that

$$\Psi_{\lambda,v} = \frac{B_c}{A_c} + \frac{2}{v_c} + O\left(\frac{1}{L^2}\right) + O\left(\frac{1}{L^2_{\lambda,v}}\right) \tag{B.23a}$$

$$V_{\lambda,v} = \frac{A_c v_c}{2\pi}L_{\lambda,v} + O\left(\frac{1}{L}\right) + O\left(\frac{1}{L_{\lambda,v}}\right). \tag{B.23b}$$

On the infinite line, as λ varies, all the eigenvectors $\phi_{\lambda,v}$ satisfy the boundary conditions at $x \to \pm\infty$. In the presence of a wall, the eigenvector $\phi_{\lambda,v}$ has to vanish at the wall, therefore the rightmost zero $x = -L - \Phi_v$ of $Q_v(x)$ must coincide with a zero $x = -nL_{\lambda,v} - \Psi_{\lambda,v}$ of $\phi_{\lambda,v}$. The boundary conditions select that way a discrete set of eigenvalues λ_n. For $v_c - v$ small, one gets then, up to exponentially small corrections in L:

$$nL_{\lambda,v} - L = \Phi_v - \Psi_{\lambda,v} \tag{B.24}$$

where n is an integer larger or equal to 1. Using the expressions (B.15, B.23a) and (B.20) this becomes

$$n\left(1 - \frac{L^2\lambda}{\pi^2}\right)^{-\frac{1}{2}} - 1 = -\frac{2}{v_c L} + O\left(\frac{1}{L^3}\right)$$

and this leads to

$$\lambda_n \simeq -\frac{(n^2-1)\pi^2}{L^2} - \frac{4n^2\pi^2}{v_c L^3} - \frac{12n^2\pi^2}{v_c^2 L^4} + O\left(\frac{1}{L^5}\right). \tag{B.25}$$

One should notice that, for $n = 1$, $\lambda_1 \sim L^{-3}$ whereas all the other eigenvalues λ_n scale as L^{-2}.

In the frame of the wall, the n-th eigenvector ϕ_n is, up to exponentially small corrections, given by:

$$\phi_n(x) \simeq \phi_{\lambda_n,v}(x + y_L)$$

and therefore in the range where $L - x \ll 1$ one has

$$\phi_n(x) \simeq \frac{A_c v_c L(-1)^{n-1}}{2n\pi}\left[\sin\left(\frac{n\pi x}{L}\right) + O\left(\frac{1}{L^2}\right)\right] e^{\frac{v_c(x-L)}{2} - \frac{v_c B_c}{2A_c}} + O\left(e^{v_c(x-L)}\right)$$

and in the range where $x - L$ is of order 1

$$\phi_n(x) \simeq Q'_{v_c}\left(x - L - \frac{B_c}{A_c}\right).$$

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